ON THE DUBINS AND SAVAGE CHARACTERIZATION OF OPTIMAL STRATEGIES

by

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1. Introduction.

An elegant characterization of optimal strategies for gambling problems in a finitely additive setting was given by Dubins and Savage in their book How to Gamble If You Must, [3]. An exposition of their ideas is presented here in a measurable, countably additive framework. With the additional measurability assumptions, it becomes possible to treat a more general class of payoff functions. Also, necessary and sufficient conditions are given for a strategy to be ϵ-optimal, a problem not considered in [3].

Most of the notation and definitions below are borrowed or adapted from [3].

2. Definitions and preliminaries.

This section establishes the framework for the sequel and reports certain technical measurability results needed there.

The term Borel set is used here to mean a Borel subset of a complete separable metric space. Let X be a Borel set. Denote by $\mathcal{B}(X)$ the Borel subsets of X and by $\mathcal{P}(X)$ the set of all countably additive probability measures on $\mathcal{B}(X)$. If $\mathcal{P}(X)$ is given the usual weak topology, then it has the structure of a Borel set and the Borel subsets of $\mathcal{P}(X)$ may be described as the smallest σ-field of subsets which makes $\gamma \to \gamma(A)$ a measurable function from $\mathcal{P}(X)$ to the Borel line for each A in $\mathcal{B}(X)$. (A thorough discussion of the weak topology is in Chapter II of [6] and the Borel structure of $\mathcal{P}(X)$ is explored in [2].)

Let F be a Borel set to be regarded as the set of fortunes of a gambler or possible states of a system. Set $\mathcal{B} = \mathcal{B}(F)$ and $\mathcal{P} = \mathcal{P}(F)$. An element γ of $\mathcal{P}$ will be called a gamble, although Dubins and Savage use that term to mean a finitely additive probability measure defined on all subsets of F. A <u>gambling house</u> Γ on F assigns to every f in F a non-void set $\Gamma(f)$ of gambles. It is assumed that the set $\{(f, \gamma): \gamma \in \Gamma(f)\}$ is a Borel subset of the product space $F \times \mathcal{P}$. The implications of such an assumption were first studied by Strauch in [7].

A <u>strategy</u> σ is a sequence $\sigma_0, \sigma_1, \ldots$, where σ_0 is a gamble and, for $n > 0$, σ_n is a measurable map from $F \times \ldots \times F$ (n-factors) into $\mathcal{P}$. Let H be the countably infinite product $F \times F \times \ldots$ with the product Borel structure. The symbol "h" will always denote a typical element or <u>history</u> $(f_1, f_2, \ldots)$ of H. A strategy σ naturally induces a probability measure $\mu(\sigma)$ on $\mathcal{B}(H)$. That is, the $\mu(\sigma)$-marginal distribution of the first coordinate f_1 is σ_0 and, given the first n coordinates are $(f_1, \ldots, f_n)$, the conditional $\mu(\sigma)$-distribution of f_{n+1} is $\sigma_n(f_1, \ldots, f_n)$. When there is no danger of confusion, we shall use the same notation σ for the strategy σ and the corresponding measure $\mu(\sigma)$.

A strategy σ is <u>essentially available at</u> f <u>in</u> Γ if $\sigma_0 \in \Gamma(f)$, and, for $n > 0$, $\sigma_n(f_1, \ldots, f_n) \in \Gamma(f_n)$ σ-almost surely. Thus the gambler must choose gambles available in the house at his current fortune almost surely at each stage of play. For $f \in F$, let $\Gamma^\infty(f)$ be the set of all strategies essentially available at f in Γ and let $\Gamma^\infty = \{(f, \mu(\sigma)): \sigma \in \Gamma^\infty(f)\}$. Then the set Γ^∞ is a Borel subset of $F \times \mathcal{P}(H)$. For a proof see Theorem 2.1 of [12] and Theorem 5.1 of [10].

A <u>partial history</u> p is a finite sequence of elements of F. If σ is a strategy and $p = (f_1, \ldots, f_n)$ is a partial history, then the <u>conditional strategy</u> $\sigma[p]$ is defined by

$$(\sigma[p])_0 = \sigma_n(f_1, \ldots, f_n)$$

and, for $k > 0$,

$$(\sigma[p])_k(f_1', \ldots, f_k') = \sigma_{n+k}(f_1, \ldots, f_n, f_1', \ldots, f_k').$$

It is easy to check that $\sigma[f_1, \ldots, f_n]$ is a version of the regular conditional σ-distribution of $(f_{n+1}, f_{n+2}, \ldots)$ given $(f_1, \ldots, f_n)$. Also, if $\sigma \varepsilon \Gamma^\infty(f)$, then $\sigma[f_1, \ldots, f_n] \varepsilon \Gamma^\infty(f_n)$ σ-almost surely.

Now let g be a measurable function from H to the extended reals. For $h \varepsilon H$, the value $g(h)$ is to be regarded as the payoff received by a gambler who experiences the history h and g is called the payoff function. To simplify the exposition, it is assumed that g is bounded above. (Many of our results would still hold if other conditions on g or on the house Γ were used to assure the existence of $\int g d\sigma$ for every available σ. The assumption here roughly corresponds to the common one of a non-negative loss function.) A gambler who plays the strategy σ has expected winnings $\int g d\sigma$. Let

$$V(f) = \sup_{\sigma \varepsilon \Gamma^\infty(f)} \int g d\sigma.$$

The function V is called the strategic utility of the house Γ and $V(f)$ may be regarded as the most that can be achieved by a gambler with fortune f. The function V is bounded above and has been shown (Theorem 5.2, [10]) to be universally measurable. (Recall that a function is universally measurable if it is measurable with respect to the completion of every measure on the Borel sets.)

3. The Dubins and Savage payoff function.

For a reader familiar with gambling theory, this section should serve as motivation for the sequel. It is not, however, a logical prerequisite for what follows.

Let u be a bounded function on F and let σ be a strategy. Dubins and Savage in [3] define the utility of σ to be

$$u(\sigma) = \limsup_{t\to\infty} \int u(f_t)d\sigma. \tag{1}$$

The lim sup is taken over the directed set of stop rules. There are no measurability requirements for u, σ, or the stop rules t. (For a definition of "strategy," "stop rule," and the integral in (1), consult [3].) If u is assumed to be $\mathcal{B}$-measurable and σ is a strategy in the sense of this note, then, by Theorem 3.2 of [10],

$$u(\sigma) = \int u^* d\sigma , \tag{2}$$

where

$$u^*(h) = \limsup_{n\to\infty} u(f_n). \tag{3}$$

Thus the problem studied by Dubins and Savage is, under our measurability assumptions,of the type described in section 2 where the payoff function g is u^*.

Now assume u to be $\mathcal{B}$-measurable and bounded above. Let σ be a strategy and define $u(\sigma)$ as in (1) except that the lim sup is to be taken over all measurable incomplete (i.e., not necessarily finite) stop rules t which are finite almost surely under σ. Then, by the Theorem of [11], formula 2 continues to hold, which is not the case if the stop rules are required to be everywhere finite as in [3].

<u>Example 1</u>: Let $F = \{-1, 1\}$ and $\sigma_0 = \sigma_n(f_1,\ldots, f_n) = \frac{1}{2}\delta(1) + \frac{1}{2}\delta(-1)$ for all $(f_1,\ldots, f_n)$. Thus σ is the distribution of a fair coin toss process with path space H. Consider also F' = set of integers and

$\sigma_0' = \sigma_0$, $\sigma_n'(f_1',\ldots, f_n') = \frac{1}{2}\delta(f_n' + 1) + \frac{1}{2}\delta(f_n' - 1)$ for all $(f_1',\ldots, f_n')$. The strategy σ' on H' is the distribution of the partial sums of a fair coin toss process. Let $u(f') = \min(f', 1)$ for all $f' \in F'$.

Suppose t' is an (everywhere finite) stop rule on H'. Let t be the stop rule on H given by

$$t(f_1, f_2, f_3,\ldots) = t'(f_1, f_1+ f_2, f_1+ f_2+ f_3,\ldots).$$ Notice

$f_1 + \ldots + f_t$ has the same distribution under σ as does $f_{t'}'$ under σ'. Now $u(f_1), u(f_1+ f_2),\ldots$ is an expectation decreasing semi-martingale under σ and, since F is finite, t is bounded (Theorem 2.9.1 of [3]). Hence,

$$0 = \int u(f_1)d\sigma \geq \int u(f_1+ \ldots + f_t)d\sigma = \int u(f_{t'}')d\sigma',$$

for every stop rule t'. A fortiori, $\limsup_{t'\to\infty} \int u(f_{t'}')d\sigma' \leq 0$. However, $u^* = 1$ σ'-almost surely, so that $\int u^* d\sigma' = 1$.

When it seems appropriate, results below will be specialized to the case when $g = u^*$ for some u and connections made with the original work in [3].

Notice that

$$u^*(f_1, f_2,\ldots) = u^*(f_2,\ldots)$$

for every $h = (f_1, f_2,\ldots)$.

4. Properties of V when the payoff function is shift-invariant.

It is assumed for the remainder that the payoff function g, in addition to being bounded above and measurable, satisfies

$$g(f_1, f_2,\ldots) = g(f_2,\ldots)$$

for every $h = (f_1, f_2,\ldots)$.

Intuitively, the gambler who has fortune f_1 after the first play should wish to play exactly as though he were entering the game with initial fortune f_1.

By the previous section, such shift-invariant payoff functions include, at least in a countably additive setting, those studied by Dubins and Savage. Moreover, many sequential optimization problems, which do not appear to have shift-invariant payoff functions, can be formulated so as to fit the model of this note.

Example 2: Suppose r is a bounded function on F, $0 < \beta < 1$, and $g(h) = \sum_{n=1}^{\infty} \beta^n r(f_n)$. Such payoff functions have been studied by Blackwell [1] and others. Of course, g is not typically shift-invariant.

Let us follow section 12.2 of [3] and set $f_n' = (f_1, \ldots, f_n)$ and $h' = (f_1', f_2', \ldots)$. Then, if $u(f_n') = \sum_{k=1}^{n} \beta^k u(f_k)$, we have $u^*(h') \equiv \limsup_{n \to \infty} u(f_n') = g(h)$, and u^* is shift-invariant on H'. Further examples and a more complete discussion are in [3].

A function Q on F is excessive for Γ if, for every $f \in F$ and $\gamma \in \Gamma(f)$, $\int Q d\gamma \leq Q(f)$.

Theorem 1: The strategic utility V is excessive for Γ.

Rather than prove Theorem 1, notice it is a special case of our next result, for whose statement we need another definition.

A stopping variable t is a measurable map defined on H, having values in the set $\{1, 2, \ldots, +\infty\}$, and such that, given $h = (f_1, f_2, \ldots)$ and $h' = (f_1', f_2', \ldots)$, if $t(h) = n$ and $f_i' = f_i$ for $1 \leq i \leq n$, then $t(h') = n$.

Theorem 2: If $f \in F$, $\sigma \in \Gamma^{\infty}(f)$, and t is a stopping variable such that $\sigma[t < +\infty] = 1$, then

$$\int V(f_t)d\sigma \leq V(f).$$

(Here, $f_t(h) = f_{t(h)}$.)

Proof: Let μ_t be the distribution of f_t under σ. Since V is measurable with respect to the completion of $\mathcal{B}$ under μ_t, there is a $\mathcal{B}$-measurable function Q such that $\mu_t\{f: Q(f) = V(f)\} = 1$ and $Q(f) \leq V(f)$ for all f in F.

Let $\varepsilon > 0$ and define

$$A = \{(f', \mu(\sigma)): \int g d\sigma \geq Q(f) - \varepsilon, \ \sigma \in \Gamma^{\infty}(f)\}.$$

Then A is a measurable subset of $F \times \mathcal{P}(H)$ and each f'-section of A is non-empty. Hence, by Theorem 6.3 of [4], there is a measurable map $\varphi: F \to \mathcal{P}(H)$ such that $\mu_t\{f': (f', \varphi(f')) \in A\} = 1$.

Now, for each $f' \in F$, choose a strategy $\bar{\sigma}(f')$ such that $\mu(\bar{\sigma}(f')) = \varphi(f')$ and choose the $\bar{\sigma}(f')$ so that $\mu_t\{f': \bar{\sigma}(f') \in \Gamma^{\infty}(f')\} = 1$.

Let σ' be the strategy which is the composition of σ with the family $\bar{\sigma}(\cdot)$ at time t. That is,

$$(\sigma')_0 = \sigma_0,$$

and, for every partial history $(f_1, \ldots, f_n)$,

$$\begin{aligned}(\sigma')_n(f_1, \ldots, f_n) &= \sigma_n(f_1, \ldots, f_n) \quad \text{if} \quad t > n, \\ &= (\bar{\sigma}(f_t))_0 \quad \text{if} \quad t = n, \\ &= (\bar{\sigma}(f_t))_{n-t}(f_{t+1}, \ldots, f_n) \quad \text{if} \quad t < n,\end{aligned}$$

where $t = t(f_1, \ldots, f_n, \ldots)$.

Then $\sigma' \in \Gamma^\infty(f)$ and

$$\int V(f_t)d\sigma = \int Qd\mu_t \leq \int \{\int gd\bar{\sigma}(f_t) + \epsilon\}d\mu_t = \int gd\sigma' + \epsilon$$

(by Fubini (Theorem II.14, [5]) and the shift-invariance of g)

$$\leq V(f) + \epsilon. \quad \square$$

Theorem 2 is essentially an optional stopping theorem and implies that $V(f_1), V(f_2), \ldots$ is a (generalized) expectation decreasing semi-martingale with respect to any $\sigma \in \Gamma^\infty(f)$. Perhaps the reader should be reminded that such an optional stopping result does not hold for arbitrary (generalized) semi-martingales which are bounded from one side.

Example 3: Let f_n, σ, and u be as in Example 1. Let $t(h) = \min\{n: f_1 + \ldots + f_n = 1\}$. Then $\sigma[t < \infty] = 1$ and $\int u(f_1 + \ldots + f_t)d\sigma = 1 > 0 = \int u(f_1)d\sigma$. It is interesting to observe that, for any stopping variable t which is finite everywhere, we do have $\int u(f_1 + \ldots + f_t)d\sigma \leq \int u(f_1)d\sigma$. A proof can be based on an obvious generalization of Theorem 2.12.1 of [3].

Corollary 1: Let $f \in F$, $\sigma \in \Gamma^\infty(f)$, and t and s be stopping variables with $\sigma[t \leq s < \infty] = 1$. Then

$$\int V(f_t)d\sigma \geq \int V(f_s)d\sigma.$$

Proof: For every $h \in H$, let $p_t(h) = (f_1, \ldots, f_{t(h)})$. Then the conditional strategies $\sigma[p_t]$ are essentially available at f_t in Γ σ-almost surely. Let $s[p_t]$ be the conditional stopping variable defined by

$$s[p_t](f_1', f_2', \ldots) = s(f_1, \ldots, f_t, f_1', f_2', \ldots).$$

Then, by Theorem 2,

$$\int V(f_{s[p_t]})d\sigma[p_t] \leq V(f_t)$$

σ-almost surely. By Fubini's Theorem,

$$\int \{V(f_t) - V(f_s)\}d\sigma = \int \{V(f_t) - \int V(f_{s[p_t]})d\sigma[p_t]\}d\sigma.$$

The desired inequality follows. □

The next result is a version of the so-called functional equation of dynamic programming.

Theorem 3: For every $f \in F$,

$$V(f) = \sup_{\gamma\in\Gamma^{\infty}(f)} \int V d\gamma.$$

Proof: Let $\varepsilon > 0$. Choose $\sigma \in \Gamma^{\infty}(f)$ such that $\int g d\sigma \geq V(f) - \varepsilon$. Then $\sigma_0 \in \Gamma(f)$ and

$$\sup_{\gamma\in\Gamma^{\infty}(f)} \int V d\gamma \geq \int V d\sigma_0 \geq \int\{\int g d\sigma[f_1]\}d\sigma_0 = \int g d\sigma \geq V(f) - \varepsilon.$$

This proves one of the needed inequalities. The other is immediate from Theorem 2. □

Now let σ be a strategy and $\{Q_n\}_{n\geq 1}$ a sequence of universally measurable functions on F which are uniformly bounded above. Define

(3) $$Q(\sigma) = \limsup_{t\to\infty} \int Q_t(f_t)d\sigma,$$

where the lim sup is over the directed set of stopping variables t such that $\sigma[t < \infty] = 1$. Also, define Q^* on H by

(4) $$Q^*(h) = \limsup_{n\to\infty} Q_n(f_n).$$

It will usually be the case below that all the Q_n are equal to some fixed function Q.

Theorem 4: If the functions Q_n are universally measurable and bounded above, then

$$Q(\sigma) = \int Q^* d\sigma.$$

Proof: If the Q_n are $\mathcal{B}$-measurable, the desired formula is a special case of the theorem in [11].

If the Q_n are universally measurable, then there is a sequence R_n of $\mathcal{B}$-measurable functions such that

$$\sigma\{h: Q_n(f_n) = R_n(f_n) \quad \text{for} \quad n = 1, 2, \ldots\} = 1.$$

The R_n may be chosen to be uniformly bounded above. Hence,

$$\int Q^* d\sigma = \int R^* d\sigma = R(\sigma) = Q(\sigma). \quad \square$$

Corollary 2: If V is the strategic utility of Γ and σ is any strategy, then

$$V(\sigma) = \int V^* d\sigma.$$

Another formula for $V(\sigma)$ is given by

Theorem 5: If V is the strategic utility of Γ and $\sigma \in \Gamma^\infty(f)$ for some $f \in F$, then

$$\int V(f_n) d\sigma \searrow V(\sigma) \quad \text{as} \quad n \to \infty.$$

Proof: By Corollary 1, if $n \geq m$, then $\int V(f_n) d\sigma \leq \int V(f_m) d\sigma$. So the limit exists.

Again by Corollary 1, if t is a stopping variable, $\sigma[t < \infty] = 1$, and $t \geq n$, then

$$\int V(f_t) d\sigma \leq \int V(f_n) d\sigma.$$

Now take the lim sup as $t \to \infty$ and then the limit as $n \to \infty$ to get

$$V(\sigma) \leq \lim_{n \to \infty} \int V(f_n) d\sigma.$$

But, by Fatou's Lemma and Corollary 2,

$$\lim_{n\to\infty} \int V(f_n)d\sigma \leq \int V^* d\sigma = V(\sigma). \square$$

The next theorem exemplifies the sort of convergence result possible if the processes under consideration are not allowed to grow too large in a negative direction.

Theorem 6: Let $f \in F$ and $\sigma \in \Gamma^\infty(f)$.

(a) If $V(\sigma) > -\infty$, then $V(f_n) \to V^*(h)$ σ-almost surely as $n \to \infty$.

(b) If $\int g d\sigma > -\infty$, then $\int g d\sigma[f_1,\ldots, f_n] \to g(h)$ σ-almost surely as $n \to \infty$ and $\sigma[g > V^*] = 0$.

Proof: (a) Since $V(f_n)$ is an expectation decreasing process by Theorem 2 and $\inf_n \int V(f_n)d\sigma = V(\sigma) > -\infty$ by Theorem 5, $V(f_n)$ converges almost surely by a supermartingale convergence theorem (Theorem V.17, [5]). The limit is, of course, V^* almost surely.

(b) By the assumption, g is integrable with respect to σ. Also, $\int g d\sigma[f_1,\ldots, f_n]$ is a version of the conditional σ-expectation of g given $(f_1,\ldots, f_n)$. Thus the convergence is by another martingale theorem (Theorem V.18, [5]).

Finally,

$$V(f_n) \geq \int g d\sigma[f_1,\ldots, f_n]$$

σ-almost surely since $\sigma[f_1,\ldots, f_n] \in \Gamma^\infty(f_n)$ σ-almost surely. Pass to the limit as $n \to \infty$ to prove the final assertion. $\square$

The final result of this section will be the starting point for the discussion of optimality here as its analogue was in [3].

Theorem 7: Let $f \in F$ and $\sigma \in \Gamma^{\infty}(f)$. Then $V(f) \geq V(\sigma) \geq \int g d\sigma$.

Proof: The first inequality is immediate from Theorem 2 and the definition of $V(\sigma)$ in (3).

To prove the second inequality, let t be a stopping variable such that $\sigma[t < \infty] = 1$. Let $p_t = (f_1, \ldots, f_t)$ and observe that $\sigma[p_t] \in \Gamma^{\infty}(f_t)$ σ-almost surely. Hence,

$$\int V(f_t) d\sigma \geq \int \{\int g d\sigma[p_t]\} d\sigma = \int g d\sigma.$$

The last equation is by Fubini's Theorem and the shift-invariance of g. Now take the lim sup as $t \to \infty$. □

5. Optimal strategies.

Let $f \in F$ and $\sigma \in \Gamma^{\infty}(f)$ be fixed for the discussion in this section.

The strategy σ is said to be optimal (ϵ-optimal) at f in Γ iff $\int g d\sigma = V(f)$ ($\int g d\sigma \geq V(f) - \epsilon$).

The strategy σ is thrifty (ϵ-thrifty) iff $V(\sigma) = V(f)$ ($V(\sigma) \geq V(f) - \epsilon$).

The strategy σ is equalizing (ϵ-equalizing) iff $\int g d\sigma = V(\sigma)$ ($\int g d\sigma \geq V(\sigma) - \epsilon$).

An immediate consequence of Theorem 7 is

Theorem 8: A strategy is optimal iff it is thrifty and equalizing. A strategy is ϵ-optimal iff it is ϵ_1-thrifty and ϵ_2-equalizing for some ϵ_1, ϵ_2 such that $\epsilon_1 + \epsilon_2 \leq \epsilon$.

Thus the notion of optimality will be characterized if we characterize thrifty strategies and equalizing strategies. The results which follow give formulae for the numbers $V(f) - V(\sigma)$ and $V(\sigma) - \int g d\sigma$, which measure the two ways in which σ may depart from optimality.

Let

$$\epsilon_0 = V(f) - \int V d\sigma_0,$$

and, for $n > 0$ and $h \in H$, let

$$\epsilon_n(h) = \epsilon_n(f_1, \ldots, f_n) = V(f_n) - \int V d\sigma_n(f_1, \ldots, f_n).$$

Notice that, by Theorem 1, $\epsilon_n \geq 0$ σ-almost surely for all n.

Theorem 9: *The strategy* σ *is* ϵ-*thrifty iff*

$$\int \left(\sum_{n=0}^{\infty} \epsilon_n\right) d\sigma \leq \epsilon.$$

In fact,

$$V(f) - V(\sigma) = \int \left(\sum_{n=0}^{\infty} \epsilon_n\right) d\sigma .$$

Proof: It is easy to see by induction on n that

$$\int V(f_n) d\sigma = V(f) - \int \left(\sum_{k=0}^{n-1} \epsilon_k\right) d\sigma.$$

Let $n \to \infty$ and the result follows from the monotone convergence theorem and Theorem 5. □

The result corresponding to Theorem 9 for a finitely additive theory of gambling is in [9].

Theorem 10: *The strategy* σ *is thrifty iff, for all* n, $\epsilon_n = 0$ σ-*almost surely*.

Proof: Immediate from Theorem 9. □

Thus a gambler is thrifty iff the $V(f_n)$ process is a (generalized) martingale with respect to his strategy. In intuitive terms, he must play in such a way that his prospective optimal winnings (i.e., his current values for V) almost never decrease in expectation.

Stated still another way, a strategy is thrifty if the gambler almost always chooses gambles γ so that $\int V d\gamma$ attains the supremum of the functional equation in Theorem 3. A common error is to assume that strategies so constructed are necessarily optimal, which brings us to equalizing strategies.

Theorem 11: *The strategy* σ *is* ϵ-*equalizing iff* $\int (V^* - g) d\sigma \le \epsilon$. *That is*,

$$V(\sigma) - \int g d\sigma = \int (V^* - g) d\sigma.$$

Proof: Immediate from Corollary 2. □

A sufficient condition for σ to be ϵ-equalizing is given next. The notation "i.o." is used below as an abbreviation for the phrase "for infinitely many n."

Theorem 12: *If* $\epsilon > 0$ *and* $\sigma\{h: V(f_n) \le g(h) + \epsilon \text{ i.o.}\} = 1$, *then* σ *is* ϵ-*equalizing*.

Proof: The condition implies $\sigma[V^* \le g + \epsilon] = 1$. Now use Theorem 11. □

The corresponding necessary condition is in

Theorem 13: *If* $\epsilon > 0$, σ *is* ϵ^2-*equalizing and* $V(\sigma) > -\infty$, *then*

$$\sigma\{h: V(f_n) \le g(h) + \epsilon \text{ i.o.}\} \ge 1 - \epsilon.$$

Proof: By assumption, $\int g d\sigma \ge V(\sigma) - \epsilon^2 > -\infty$. So, by Theorem 6(b), $V^* - g \ge 0$ σ-almost surely. Also, by Theorem 11, $\int (V^* - g) d\sigma \le \epsilon^2$. Hence, $\sigma\{h: V^*(h) - g(h) < \epsilon\} \ge 1 - \epsilon$. The conclusion now follows from the definition of V^* as $\limsup_{n\to\infty} V(f_n)$. □

The previous two theorems imply

Theorem 14: *Suppose* $V(\sigma) > -\infty$. *Then* σ *is equalizing iff, for all* $\epsilon > 0$,

$$\sigma\{h: V(f_n) \le g(h) + \epsilon \text{ i.o.}\} = 1.$$

Let us now assume that u is a $\mathcal{B}$-measurable function from F to the reals which is bounded above and consider the associated Dubins and Savage payoff function u^*. It is of some interest to reinterpret our results for such a payoff function and thus make explicit contact with the original work in [3].

<u>Theorem 15</u>: <u>Let</u> $g = u^*$ <u>and suppose</u> $\int u^* d\sigma > -\infty$. <u>Then the following are equivalent</u>:

(a) σ is equalizing;

(b) $u(\sigma) = V(\sigma)$;

(c) for all $\epsilon > 0$, $\sigma\{h: u^*(h) \geq V(f_n) - \epsilon \text{ i.o.}\} = 1$;

(d) for all $\epsilon > 0$, $\sigma\{h: u(f_n) \geq V(f_n) - \epsilon \text{ i.o.}\} = 1$;

(e) for all $\epsilon > 0$, $v_\epsilon(\sigma) = 1$, where v_ϵ is the indicator function of the set $\{f: u(f) \geq V(f) - \epsilon\}$.

<u>Proof</u>: Since $u(\sigma) = \int u^* d\sigma$, (a) and (b) are equivalent. Also, (a) and (c) are equivalent by the previous theorem; and (d) and (e) are easily seen to be equivalent if Q is set equal to v_ϵ in Theorem 4. It will be enough to show (c) and (d) are equivalent.

Let $\{x_n\}$ and $\{y_n\}$ be sequences of real numbers and let $x^* = \limsup_{n\to\infty} x_n$. The following implications can be checked for any $\epsilon > 0$:

(i) if $x_n \geq y_n - \epsilon$ i.o., then $x^* \geq y_n - 2\epsilon$ i.o.;

(ii) if $x^* \geq y_n - \epsilon$ i.o. and y_n converges to a finite limit, then $x_n \geq y_n - 2\epsilon$ i.o.

By (i), we see that (d) $\Rightarrow$ (c). By Theorem 6(a), $V(f_n)$ converges σ-almost surely to V^*, so that the opposite implication follows from (ii). □

Condition (b) was used in [3] to define equalizing and was shown there to be equivalent to (e). Theorems 13 and 14 can also be reinterpreted for $g = u^*$.

In summation, a strategy is optimal iff it is thrifty and equalizing. It is thrifty iff the expected value of V does not decrease and equalizing iff the gambler is almost sure to reach fortunes whose utility is almost V infinitely often.

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